From (5) and (1) it is seen that $A=3$, $B=-\frac{1}{3}$, $C=3$.

$\therefore x=3a$, $y=-\frac{1}{3}b$, $z=3C$.

In like manner, $x=3a$, $y=3b$, $z=-\frac{1}{3}c$, and $x=-\frac{1}{3}a$, $y=3b$, $z=3c$.

Also solved by M. E. Graber, Grace M. Bareis, E. L. Sherwood, Christian Hornung, F. P. Matz, G. B. M. Zerr, and the Proposer.

216. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Express by radicals the roots of $x^6+ax^4+bx^3+\frac{1}{4}a^2x^2+\frac{1}{2}abx+c=0$.

I. Solution by E. L. SHERWOOD.

$x^6+ax^4+\frac{1}{4}a^2x^2+bx^3+\frac{1}{2}abx+c=0$, $(x^3+\frac{1}{2}ax)^2+b(x^3+\frac{1}{2}ax)+c=0$, whence we have, by solving the quadratic

$$x^3+\tfrac{1}{2}ax+\frac{2b-b^2+4c}{4}=0, \text{ or } x^3+\tfrac{1}{2}ax+\frac{2b+b^2-4c}{4}=0,$$

whence by Cardan's method, Burnside and Panton, p. 108,

$$x=\sqrt[3]{p}+\frac{-H}{\sqrt[3]{p}},\quad \omega\sqrt[3]{p}-\frac{H}{\omega\sqrt[3]{p}},\quad \omega^2\sqrt[3]{p}-\frac{H}{\omega^2\sqrt{p}}$$

where $p=\frac{1}{2}[\sqrt{(G^2+4H^3)}-G]$, and $G=\dfrac{2b-b^2+4c}{4}$ or $\dfrac{2b+b^2-4c}{4}$, $H=\frac{1}{6}a$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Write the equation as follows:

$$x^2(x^2+\tfrac{1}{2}a)^2+bx(x^2+\tfrac{1}{2}a)+c=0.$$

Let $x(x^2+\frac{1}{2}a)=x^3+\frac{1}{2}ax=y$. $\quad\therefore y^2+by+c=0$.

$\therefore y=\dfrac{-b\pm\sqrt{(b^2-4c)}}{2}$.

Let ω be an imaginary cube root of unity, and let m, n be the roots of $t^2-yt-a^3/216=0$.

$\therefore x=m+n$, $x=\omega m+\omega^2 n$, $x=\omega^2 m+\omega n$.

As y has two values, the six values of x are expressed as radicals.

Also solved by J. Scheffer, G. W. Greenwood, Elmer Schuyler, F. P. Matz, and the Proposer.

217. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the condition that $E\equiv x^5-bx^3+cx^2+dx-e$ shall be the product of a complete square and a complete cube.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The factors must be of the form

$$(x^2-2ax+a^2)(x^3+2ax^2+\frac{4a^2x}{3}+\frac{8a^3}{27}),$$

hence $b=\frac{5}{3}a^2$, $c=-\frac{10}{27}a^3$, $d=\frac{20}{27}a^4$, $e=\frac{8}{27}a^5$, and therefore

$$\sqrt{\frac{3b}{5}}=-3\sqrt[3]{\frac{c}{10}}=\sqrt[4]{\frac{27d}{20}}=\sqrt[5]{\frac{27e}{8}}.$$

II. Solution by E. L. RICH, Lehigh University.

Consider the cube, $x^3+3px^2+3p^2x+x^3$, and the square, $x^2+2qx+q^2$. Their product is,

$$x^5+(2q+3p)x^4+(q^2+3p^2+6pq)x^3+(3pq^2+6p^2q+p^3)x^2+(2p^3q+3p^2q^2)x+p^3q^2.$$

Then the first condition is $2q+3p=0$ or $q=-\frac{3}{2}p$. The other conditions for the values of the coefficients gotten by equating coefficients, and substituting the first condition are,

$$b=\tfrac{15}{4}p^2,\ c=-\tfrac{5}{4}p^3,\ d=\tfrac{15}{4}p^4,\ e=-\tfrac{9}{4}p^5.$$

Also solved by J. Scheffer, G. B. M. Zerr, Elmer Schuyler, and F. P. Matz.

218. Proposed by SAUL EPSTEEN, The University of Chicago, Chicago, Ill.

Prove that $\sum\limits_{r=0}^{n}\frac{c_r}{r+1}=\frac{2^{n+1}-1}{n+1}$ where c_r is the coefficient of x^r in the expansion of $(1+x)^n$.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), and HOWARD M. ARMSTRONG.

$$\sum_{r=0}^{r=n}\frac{c_r}{r+1}=\int_0^1[c_0+c_1x+c_2x^2+\dots\dots+c_nx^n]dx=\int_0^1(1+x)^ndx=\frac{2^{n+1}-1}{n+1}.$$

II. Solution by J. SCHEFFER, Kee Mar College.

By the binomial theorem

$$2^{n+1}-1=(1+1)^{n+1}-1=(n+1)+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\dots\dots$$

$$\therefore\ \frac{2^{n+1}-1}{n+1}=1+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\dots\dots=1+\tfrac{1}{2}\binom{n}{1}+\tfrac{1}{3}.\binom{n}{2}+\dots=\sum_{r=0}^{n}\frac{c_r}{n+1}.$$

III. Solution by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

$$\sum_{r=0}^{n}\frac{c_r}{r+1}=c_0+\tfrac{1}{2}c_1+\tfrac{1}{3}c_2+\tfrac{1}{4}c_3+\dots\dots\frac{1}{r+1}+c_r\dots+\frac{1}{n+1}c_n$$

$$=1=\tfrac{1}{2}n+\tfrac{1}{3}.\frac{n(n-1)}{2!}+\tfrac{1}{4}.\frac{n(n-1)(n-2)}{3!}+\dots\dots\frac{1}{n+1}$$

$$=\frac{1}{n+1}(n+1+\tfrac{1}{2}n(n+1)+\tfrac{1}{3}.\frac{n(n-1)(n+1)}{2!}+\dots\dots+1)$$